

Linearized Modes in Extended and Critical Gravities

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ABSTRACT

We construct explicit solutions for the linearized massive and massless spin-2, vector and scalar modes around the AdS spacetimes in diverse dimensions. These modes may arise in extended (super)gravities with higher curvature terms in general dimensions. Log modes in critical gravities can also be straightforwardly deduced. We analyze the properties of these modes and obtain the tachyon-free condition, which allows negative mass square for these modes. However, such modes may not satisfy the standard AdS boundary condition and can be truncated out from the spectrum.

1 Introduction

One natural and simple method of resolving the renormalizability of General Relativity is to extend the Einstein theory with higher-order curvature terms; however, the procedure tends to introduce ghost massive spin-2 modes, in addition to the usual massless graviton [1, 2]. The situation is improved in $D = 3$, since the massless graviton is pure gauge and hence the theory can become unitary by reversing the sign of the Einstein-Hilbert action. The examples include the celebrated topologically massive gravity [3] and the recently constructed new massive gravity [4]. When the cosmological constant is included, there can exist a critical point in the parameter space for which the unitarity can be achieved without having to reverse the sign of the Einstein-Hilbert action [5].

With recent interest and progress in three dimensions, it is natural to extend the discussion to $D = 4$. Critical gravity in four dimensions was proposed in [6]. It improves significantly the situation discussed in [1, 2] with the introduction of a cosmological constant. The theory contains the Einstein Hilbert action, the cosmological constant and a Weyl-square term. It is however somewhat vacuous, in that many physical quantities, such as the energy of the massless graviton, mass and entropy of the Schwarzschild-AdS (Anti-de Sitter) black hole, vanish identically at the critical point. These results are consistent with the latest proposal [7] that cosmological Einstein gravity can emerge from conformal gravity, which involves only the Weyl-square term, in the long-wavelength limit precisely at the critical point. In a recent work [8], it was shown that the relation between Einstein and conformal gravities can be extended to six dimensions, and hence likely in general even dimensions. Furthermore, the unitarity in the Weyl-square extended gravity [6] can be achieved in a wider continuous parameter region than just at the critical point [8]. This provides a much more likely candidate for renormalizable quantum gravity. The supersymmetric generalization of the critical gravity and its unitary extension were obtained in [9], analogous to the off-shell extended supergravities in three dimensions [10]-[14]. Recent related works in critical gravities can be found in [15]-[27]

Many important properties derived in extended gravities rely on the explicit solutions of the massive and massless graviton modes. For example, once we know the explicit form of a mode, we can determine whether it is a ghost or not by evaluating its energy. We can also determine whether the mode is consistent with the AdS boundary condition. It turns out that the tachyon-free condition, *i.e.* the absence of exponential growth in time, allows negative mass square in AdS spacetime even for spin-2 modes, generalizing the Breitenlohner-Freedman bound for scalars. However, these modes have a slower falloff

at the AdS boundary. One can then engineer the parameters such that the inevitable ghosts are these massive modes, which can be truncated out by the standard boundary condition, leaving only the massless graviton in the spectrum, which can be evaluated to have positive energy [8]. In this analysis, the explicit solutions of the massive and massless graviton modes play an important role. For future research in extended gravities, it is therefore advantageous to construct the linearized modes around the AdS spacetimes in general dimensions. The task is made simpler by the fact that in many examples of extended gravities, the differential operators acting on the modes factorize and hence the general solutions are simply the linear combinations of massless and massive modes with different masses.

In this paper, we present linearized modes around the AdS vacua that could arise in extended gravities in diverse dimensions. (Examples of full non-linear pp-wave type of solutions in extended gravities can be found in [18, 19].) In general, the spectrum contains the massive trace scalar, massless and massive spin-2 modes. The vector modes from the metric can be removed by the diffeomorphism which is preserved in extended gravities. Around the AdS vacuum, the equation of motion for the trace scalar mode, in a suitable gauge choice, involves two derivatives less than the spin-2 modes [6], and hence they are undesirable from the point of view of renormalizability. (This conclusion differs from that in Minkowskian vacuum obtained in [1, 2].) Thus the trace mode in critical gravity is typically decoupled from the system by a suitable parameter choice [6]. (See also [4].) If the trace mode were present, it could provide a source for the spin-2 modes in higher-derivative theories, even at the linearized level. Although a field redefinition of the spin-2 modes can remove the source [11], it involves derivatives and hence may not be invertible. In this paper, we shall only consider the situation where the spin-2 modes are decoupled from the trace mode. In this case, the most general solutions are simply the linear combination of massless and massive graviton modes (or log modes in the critical points). Note that the (spin-0) scalar and vector (spin-1) matter fields do however arise in Weyl-squared supergravity [9]. Thus in this paper, we shall present the explicit construction of the general linearized decoupled massive (massless) scalar, vector and spin-2 graviton modes around the AdS vacua in diverse dimensions.

In section 2, we present the general formalism for the construction. The construction, which follows from [5, 20], is based on the fact that the AdS_D spacetime has $SO(2, D-1)$ isometry. We can hence obtain the general modes by obtaining the highest weight state of the $SO(2, D-1)$. The full representation can be obtained by the action of the creation

generators of the $SO(D-1)$ subgroup. In section 3, we obtain the explicit results for the scalar, vector and spin-2 modes in $D=3$ and $D=4$. We recover the previously known spin-2 solutions in $D=3$ [5] and $D=4$ [20]. In section 4, we present the explicit new solutions for AdS_5 and AdS_6 . The general solutions in an arbitrary dimension are presented in section 5. In section 6, we give the general formalism on how to obtain log modes at the critical point from the solutions of general massive modes. The conclusion and further discussions are given in section 7, where we give a definition of the “true mass” of linearized modes in AdS backgrounds.

2 General formalism

The D -dimensional anti-de Sitter spacetime (AdS_D) can be represented as its embedding in the $(D+1)$ -dimensional flat space as a hyperboloid

$$x_1^2 + x_2^2 - \sum_{i=3}^D x_i^2 = L^2, \quad (1)$$

with the flat metric

$$ds^2 = -dx_1^2 - dx_2^2 + \sum_{i=3}^D dx_i^2. \quad (2)$$

The global coordinates of the AdS spacetime is given by

$$\begin{aligned} x_1 &= L \cosh \rho \cos \tau, & x_2 &= L \cosh \rho \sin \tau, \\ x_{2a+1} &= L \sinh \rho u_a \cos \phi_a, & x_{2a+2} &= L \sinh \rho u_a \sin \phi_a, \end{aligned} \quad (3)$$

where L is the “radius” of the AdS, and the index a runs from 1 to $[\frac{D}{2}]$, and

$$\sum_{a=1}^{[\frac{D}{2}]} u_a^2 = 1. \quad (4)$$

Note that for odd D , the ϕ coordinate with the largest index should be set to zero. The AdS_D metric in the global coordinates is given by

$$ds^2 = L^2 \left[-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho \left(\sum_a^{[\frac{D}{2}]} du_a^2 + \sum_a^{[\frac{D-1}{2}]} u_a^2 d\phi_a^2 \right) \right], \quad (5)$$

The metric is Einstein and maximally symmetric, namely

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \quad R_{\mu\nu\rho\sigma} = \frac{\Lambda}{D-1} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \quad (6)$$

with the cosmological constant $\Lambda = -(D-1)L^2$.

The AdS_D spacetime possesses the isometry group $SO(2, D-1)$, generated by the following Killing vectors

$$\begin{aligned} L_{12} &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}, & L_{ij} &= x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \\ L_{1i} &= -i \left(x_1 \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_1} \right), & L_{2i} &= -i \left(x_2 \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_2} \right). \end{aligned} \quad (7)$$

These Killing vectors generate the following algebra

$$[L_{ij}, L_{kl}] = -\delta_{ik} L_{jl} - \delta_{jl} L_{ik} + \delta_{il} L_{jk} + \delta_{jk} L_{il}. \quad (8)$$

Note that in the above the indices include the 1, 2 directions as well.

Let us classify the Cartan and root generators. When D is odd, namely $D = 2n - 1$. The group $SO(2, 2n - 2)$ belongs to Lie group D_n . The root vectors for the simple roots are given by

$$\begin{aligned} \vec{\alpha}_i &= \vec{e}_i - \vec{e}_{i+1} & 1 \leq i \leq n-1, \\ \vec{\alpha}_n &= \vec{e}_{n-1} + \vec{e}_n, \end{aligned} \quad (9)$$

where $\vec{e}_1 = (-1, 0, \dots, 0)$ and $\vec{e}_i = (0, \dots, 1, \dots, 0)$ for $i \neq 1$. Note that the unusual sign choice for \vec{e}_1 , which is convenient for the non-compact group. A natural choice of the Cartan generators for our AdS metrics in global coordinates is given by

$$H_i = i L_{(2i-1)(2i)}, \quad i = 1, 2, \dots, n. \quad (10)$$

The corresponding simple roots are then given by

$$\begin{aligned} E_{\vec{\alpha}_1} &= \frac{1}{2} (L_{13} + i L_{14} + i L_{23} - L_{24}), \\ E_{\vec{\alpha}_i} &= \frac{1}{2} (L_{(2i-1)(2i+1)} + i L_{(2i-1)(2i+2)} - i L_{(2i)(2i+1)} + L_{(2i)(2i+2)}), \quad 2 \leq i \leq n-1, \\ E_{\vec{\alpha}_n} &= \frac{1}{2} (L_{(2n-3)(2n-1)} - i L_{(2n-3)(2n)} - i L_{(2n-2)(2n-1)} - L_{(2n-2)(2n)}). \end{aligned} \quad (11)$$

It is easy to verify that they satisfy

$$[H_i, H_j] = 0, \quad [H_i, E_{\vec{\alpha}_x}] = \vec{\alpha}_x^i E_{\vec{\alpha}_j}, \quad x = 1, \dots, n \quad (12)$$

$$[E_{\vec{\alpha}_x}, E_{-\vec{\alpha}_x}] = \frac{2}{|\vec{\alpha}_x|^2} \vec{\alpha}_x^i H_i, \quad (13)$$

where

$$|\vec{\alpha}_x|^2 = \sum_{i=1}^n (\vec{\alpha}_x^i)^2. \quad (14)$$

All the negative (simple) roots are then given by $E_{-\vec{\alpha}_x} = \pm (E_{\vec{\alpha}_x})^*$. (Note that the specific choice of the \pm signs in the negative roots are determined according to the algebraic relation (13).)

For $D = 2n$, the isometry group $SO(2, 2n-1)$ belongs to the B_n series. The root vectors for the simple roots are

$$\begin{aligned}\vec{\alpha}_i &= \vec{e}_i - \vec{e}_{i+1}, \quad 1 \leq i \leq n-1, \\ \vec{\alpha}_n &= \vec{e}_n.\end{aligned}\tag{15}$$

The Cartan generators and the positive roots are chosen to be

$$H_i = i L_{(2i-1)(2i)}, \quad i = 1, 2, \dots, n,\tag{16}$$

and

$$\begin{aligned}E_{\vec{\alpha}_1} &= \frac{1}{2} (L_{13} + i L_{14} + i L_{23} - L_{24}), \\ E_{\vec{\alpha}_i} &= \frac{1}{2} (L_{(2i-1)(2i+1)} + i L_{(2i-1)(2i+2)} - i L_{(2i)(2i+1)} + L_{(2i)(2i+2)}), \quad 2 \leq i \leq n-1, \\ E_{\vec{\alpha}_n} &= L_{(2n-1)(2n+1)} - i L_{(2n)(2n+1)}.\end{aligned}\tag{17}$$

They also satisfy the commutation relations (12), (13).

In this paper, we are looking for solutions that are eigenstates of Cartan generators and annihilated by all the positive-root generators, namely

$$\begin{aligned}H_1|\psi\rangle &= E_0|\psi\rangle, \quad H_i|\psi\rangle = s_{i-1}|\psi\rangle \quad \text{for } i = 2, 3, \dots, \\ E_{\vec{\alpha}_x}|\psi\rangle &= 0, \quad \text{for all simple roots } \alpha_x \text{ and hence all positive roots.}\end{aligned}\tag{18}$$

Note that the generators act on the solution as Lie derivatives. Here ψ includes the scalar Φ , vector A_μ and spin-2 modes $h_{\mu\nu}$, satisfying

$$(\square - M_0^2)\Phi = 0, \quad (\square + \frac{1}{L^2} - M_1^2)A_\mu = 0, \quad (\square + \frac{2}{L^2} - M_2^2)h_{\mu\nu} = 0.\tag{19}$$

The vector A_μ satisfies the transverse condition

$$\nabla^\mu A_\mu = 0,\tag{20}$$

and $h_{\mu\nu}$ is both traceless and transverse

$$g^{\mu\nu}h_{\mu\nu} = 0, \quad \nabla^\mu h_{\mu\nu} = 0.\tag{21}$$

The reason we consider the vector modes in this paper is that massive vectors naturally appear in critical supergravities in the Proca form, satisfying $d*F = c*A$. The transverse condition (20) follows straightforwardly. (See [9].)

The equations in (19) can be summarized in the unified form for all spin- s fields

$$(\square + \frac{s}{L^2} - M_s^2)\psi_s = 0.\tag{22}$$

Not that in this formalism, the Casimir operator \mathcal{E} is related to the covariant Laplacian operator Δ as follows

$$\mathcal{E} = \sum_i H_i H_i + \sum_x \frac{1}{2} |\alpha_x|^2 (E^{\alpha_x} E^{-\alpha_x} + E^{-\alpha_x} E^{\alpha_x}) = -L^2 \Delta, \quad (23)$$

where two sums are over all the Cartan and root generators respectively. Thus the solutions of (18) must also satisfy the equations in (19), with appropriate M_0 , M_1 and M_2 .

Having obtained the highest weight state, we can obtain the remaining modes by acting on the state with the negative root generators in the subgroup $SO(D-1)$ of the $SO(2, D-1)$. Thus the massive modes with the same E_0 and hence the same M_s^2 form scalar, vector and spin-2 representations of $SO(D-1)$, corresponding to have 1, $D-1$ and $\frac{1}{2}D(D-1)-1$ degrees of freedom respectively. In the massless limit, the scalar, vector and spin-2 modes form representations of $SO(D-2)$, corresponding to $D-2$ and $\frac{1}{2}(D-1)(D-2)-1$ degrees of freedom, owing to the additional gauge symmetry. Note that this procedure for getting the linearized spin-2 modes were spelled out in [5] for three-dimensions and in [20] for four-dimensions.

It should be pointed out that M_s^2 defined in (22) is only the mass parameter. To determine the true mass of a mode, it is necessary to determine first the proper definition of the masslessness. For the spin-2 graviton, it follows from the Einstein theory that the massless graviton corresponds indeed to $M_2 = 0$. The massless vector ($s = 1$) can be defined that it preserves the gauge symmetry, and hence $d*dA = 0$, corresponding to

$$(\square + \frac{D-1}{L^2})A_\mu^{(0)} = 0. \quad (24)$$

The masslessness of a scalar field in AdS is more subtle, since there is no difference from the point of view of symmetry between the massive and massless scalars. However, as we shall see in section 7, by studying the falloff behavior, a massless scalar can be nevertheless defined.

In order to find the solutions in global coordinates, we further transform the generators H_i and $E_{\tilde{\alpha}_x}$ into the expressions in terms of the global coordinates of the AdS metric (5). We shall give the explicit expressions of these generators in the low-lying dimensions in the follow sections.

3 Linearized modes in AdS₃ and AdS₄

3.1 AdS₃

The AdS₃ metric is given by setting $u_1 = 1$ in the general AdS metric (5), namely

$$ds^2 = L^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \right). \quad (25)$$

The isometry group of the AdS₃ is $SO(2, 2)$. There are two Cartan generators

$$H_1 = iL_{12} = i\frac{\partial}{\partial\tau}, \quad H_2 = iL_{34} = i\frac{\partial}{\partial\phi}. \quad (26)$$

The simple roots are $\vec{\alpha}_1 = (-1, -1)$, $\vec{\alpha}_2 = (-1, 1)$, and the corresponding generators are

$$\begin{aligned} E_{\vec{\alpha}_1} &= \frac{1}{2} (L_{13} + iL_{14} + iL_{23} - L_{24}) \\ &= \frac{1}{2} e^{i(\tau+\phi)} \tanh \rho \frac{\partial}{\partial\tau} - \frac{1}{2} i e^{i(\tau+\phi)} \frac{\partial}{\partial\rho} + \frac{1}{2} e^{i(\tau+\phi)} \coth \rho \frac{\partial}{\partial\phi}, \\ E_{\vec{\alpha}_2} &= \frac{1}{2} (L_{13} - iL_{14} + iL_{23} + L_{24}) \\ &= \frac{1}{2} e^{i(\tau-\phi)} \tanh \rho \frac{\partial}{\partial\tau} - \frac{1}{2} i e^{i(\tau-\phi)} \frac{\partial}{\partial\rho} - \frac{1}{2} e^{i(\tau-\phi)} \coth \rho \frac{\partial}{\partial\phi}. \end{aligned} \quad (27)$$

The corresponding negative roots are given by

$$E_{-\vec{\alpha}_1} = (E_{\vec{\alpha}_1})^*, \quad E_{-\vec{\alpha}_2} = (E_{\vec{\alpha}_2})^*. \quad (28)$$

Scalar modes:

We look for a scalar mode $\Phi(\tau, \rho, \phi)$ that forms the highest weight state of the $SO(2, 2)$ algebra. This is an eigenstate of the Cartan generators H_1 and H_2 , with eigenvalues E_0 and s , *i.e.*

$$H_1 \Phi = E_0 \Phi, \quad H_2 \Phi = s \Phi, \quad (29)$$

while it is annihilated by the positives root generators $E_{\vec{\alpha}_x}$

$$E_{\vec{\alpha}_x} \Phi = 0, \quad x = 1, 2. \quad (30)$$

We find that a nontrivial solution arises only when $s = 0$; it is given by

$$\Phi = e^{-iE_0\tau} (\cosh \rho)^{-E_0}, \quad (31)$$

It is easy to verify that the Laplacian action on Φ

$$\Delta \Phi = -\square \Phi = -\nabla_\mu \nabla^\mu \Phi = -\frac{E_0(E_0 - 2)}{L^2} \Phi. \quad (32)$$

Vector modes:

The vector modes $A_\mu(\tau, \rho, \phi)$ that we are looking for also belong to the highest weight state of $SO(2, 2)$ algebra, defined by

$$\begin{aligned} H_1 A_\mu &= E_0 A_\mu, & H_2 A_\mu &= s A_\mu, \\ E_{\vec{\alpha}_x} A_\mu &= 0, & x &= 1, 2. \end{aligned} \quad (33)$$

We find that non-trivial solution may arise when $s = \pm 1$, given by

$$\begin{aligned} A_\tau &= e^{-i(E_0 \tau \pm \phi)} (\cosh \rho)^{-E_0} \sinh \rho, \\ A_\rho &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} A_\tau, \\ A_\phi &= \pm A_\tau, \end{aligned} \quad (34)$$

The gauge condition (20) is automatically satisfied. The box and Laplacian acting on the solutions are given by

$$\square A_\mu = \frac{E_0^2 - 2E_0 - 1}{L^2} A_\mu, \quad (35)$$

$$\Delta A_\mu = -\square A_\mu + R_{\mu\nu} A^\nu = -\frac{(E_0 - 1)^2}{L^2} A_\mu. \quad (36)$$

Spin-2 modes:

The spin-2 modes $\psi_{\mu\nu}(\tau, \rho, \phi)$ that belong to the highest weight state of $SO(2, 2)$ algebra are defined by

$$\begin{aligned} H_1 \psi_{\mu\nu} &= E_0 \psi_{\mu\nu}, & H_2 \psi_{\mu\nu} &= s \psi_{\mu\nu}, \\ E_{\vec{\alpha}_x} \psi_{\mu\nu} &= 0, & x &= 1, 2. \end{aligned} \quad (37)$$

The modes also satisfy the traceless and transverse conditions

$$g^{\mu\nu} \psi_{\mu\nu} = 0, \quad \nabla^\mu \psi_{\mu\nu} = 0. \quad (38)$$

Non-trivial solutions arise when $s = \pm 2$, given by

$$\begin{aligned} \psi_{\tau\tau} &= e^{-i(E_0 \tau \pm 2\phi)} (\cosh \rho)^{-E_0} (\sinh \rho)^2, \\ \psi_{\tau\rho} &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} \psi_{\tau\tau}, \\ \psi_{\tau\phi} &= \pm \psi_{\tau\tau}, & \psi_{\rho\rho} &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} \psi_{\tau\rho}, \\ \psi_{\rho\phi} &= \pm \psi_{\tau\rho}, & \psi_{\phi\phi} &= \psi_{\tau\tau}. \end{aligned} \quad (39)$$

The traceless and transverse conditions (38) are automatically satisfied. The box and Laplacian operators acting on the solutions are given by

$$\begin{aligned}\square\psi_{\mu\nu} &= \frac{E_0^2 - 2E_0 - 2}{L^2}\psi_{\mu\nu}, \\ \Delta\psi_{\mu\nu} &= -\square\psi_{\mu\nu} - 2R^\rho_{\mu\sigma\nu}\psi_\rho{}^\sigma + 2R_{(\mu}{}^\rho\psi_{\nu)\rho} \\ &= -\frac{E_0^2 - 2E_0 + 4}{L^2}\psi_{\mu\nu}.\end{aligned}\tag{40}$$

These spin-2 modes are precisely the ones given in [5], with $E_0 = h + \bar{h}$ and $s = h - \bar{h}$, where the $SO(2, 2)$ is interpreted as $SL(2, R) \times SL(2, R)$.

Note that we have followed the convention of [5] that the complex solution for the spin-2 modes is denoted as $\psi_{\mu\nu}$, with the understanding that the $h_{\mu\nu}$, which must be real, is the real or imaginary part of $\psi_{\mu\nu}$. However, we use the same notation Φ and A_μ to denote the corresponding complex solutions to economize the notations.

3.2 AdS₄

Letting $u_1 = \sin\theta$, $u_2 = \cos\theta$ in the general metric (5), we have the standard AdS₄ metric in global coordinates

$$ds^2 = L^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) \right). \tag{41}$$

The isometry group is $SO(2, 3)$. There are two Cartan generators

$$H_1 = iL_{12} = i\frac{\partial}{\partial\tau}, \quad H_2 = iL_{34} = i\frac{\partial}{\partial\phi}. \tag{42}$$

The two simple roots are $\vec{\alpha}_1 = (-1, -1)$, $\vec{\alpha}_2 = (0, 1)$, and the remaining two positive roots $\vec{\alpha}_3 = (-1, 0)$, $\vec{\alpha}_4 = (-1, 1)$. The corresponding generators are

$$\begin{aligned}E_{\vec{\alpha}_1} &= \frac{1}{2} (L_{13} + iL_{14} + iL_{23} - L_{24}) \\ &= \frac{1}{2} e^{i(\tau+\phi)} \sin\theta \tanh\rho \frac{\partial}{\partial\tau} - \frac{1}{2} i e^{i(\tau+\phi)} \sin\theta \frac{\partial}{\partial\rho} \\ &\quad - \frac{1}{2} i e^{i(\tau+\phi)} \cos\theta \coth\rho \frac{\partial}{\partial\theta} + \frac{1}{2} e^{i(\tau+\phi)} \csc\theta \coth\rho \frac{\partial}{\partial\phi}, \\ E_{\vec{\alpha}_2} &= L_{35} - iL_{45} \\ &= -e^{-i\phi} \frac{\partial}{\partial\theta} + i e^{-i\phi} \cot\theta \frac{\partial}{\partial\phi}, \\ E_{\vec{\alpha}_3} &= L_{15} + iL_{25} \\ &= e^{i\tau} \cos\theta \tanh\rho \frac{\partial}{\partial\tau} - i e^{i\tau} \cos\theta \frac{\partial}{\partial\rho} + i e^{i\tau} \sin\theta \coth\rho \frac{\partial}{\partial\theta}, \\ E_{\vec{\alpha}_4} &= \frac{1}{2} (L_{13} - iL_{14} + iL_{23} + L_{24}) \\ &= \frac{1}{2} e^{i(\tau-\phi)} \sin\theta \tanh\rho \frac{\partial}{\partial\tau} - \frac{1}{2} i e^{i(\tau-\phi)} \sin\theta \frac{\partial}{\partial\rho} \\ &\quad - \frac{1}{2} i e^{i(\tau-\phi)} \cos\theta \coth\rho \frac{\partial}{\partial\theta} - \frac{1}{2} e^{i(\tau-\phi)} \csc\theta \coth\rho \frac{\partial}{\partial\phi}.\end{aligned}\tag{43}$$

Note that $[E_{\vec{\alpha}_1}, E_{\vec{\alpha}_2}] = E_{\vec{\alpha}_3}$, $[E_{\vec{\alpha}_2}, E_{\vec{\alpha}_3}] = 2E_{\vec{\alpha}_4}$. All the negative-root generators are given by

$$E_{-\vec{\alpha}_1} = (E_{\vec{\alpha}_1})^* , \quad E_{-\vec{\alpha}_2} = -(E_{\vec{\alpha}_2})^* , \quad E_{-\vec{\alpha}_3} = (E_{\vec{\alpha}_3})^* , \quad E_{-\vec{\alpha}_4} = (E_{\vec{\alpha}_4})^* . \quad (44)$$

The generators for the $SO(3)$ subgroup of the $SO(2, 3)$ are $H_2, E_{\vec{\alpha}_2}$ and $E_{-\vec{\alpha}_2}$.

Scalar modes:

The scalar mode associated with the highest weight state of $SO(2, 3)$ has $s = 0$ and it is given by

$$\Phi = e^{-iE_0\tau} (\cosh \rho)^{-E_0} . \quad (45)$$

The Laplacian action on Φ is

$$\Delta \Phi = -\square \Phi = -\nabla_\mu \nabla^\mu \Phi = -\frac{E_0(E_0 - 3)}{L^2} \Phi . \quad (46)$$

It is clear that the scalar is a singlet under the $SO(3)$ subgroup.

Vector modes:

The non-trivial solution has $s = 1$, given by

$$\begin{aligned} A_\tau &= e^{-i(E_0\tau + \phi)} \sin \theta (\cosh \rho)^{-E_0} \sinh \rho , \\ A_\rho &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} A_\tau , \\ A_\theta &= i (\sin \theta)^{-1} (\cos \theta) A_\tau , \\ A_\phi &= A_\tau . \end{aligned} \quad (47)$$

The gauge condition (20) is automatically satisfied. The action of the box and Laplacian operators on the solution is given by

$$\square A_\mu = \frac{E_0^2 - 3E_0 - 1}{L^2} A_\mu , \quad \Delta A_\mu = -\frac{(E_0 - 1)(E_0 - 2)}{L^2} A_\mu . \quad (48)$$

Acting repeatedly on the highest-weight state ($s = 1$) with the creation generator $E_{-\vec{\alpha}_2}$, we obtain the $s = 0, -1$ modes as well. Together, they form the spin-1 representation of $SO(3)$.

Spin-2 modes:

In this case, we must have $s = 2$ and the solution is given by

$$\psi_{\tau\tau} = e^{-i(E_0\tau + 2\phi)} (\sin \theta)^2 (\cosh \rho)^{-E_0} (\sinh \rho)^2 ,$$

$$\begin{aligned}
\psi_{\tau\rho} &= i(\cosh\rho)^{-1}(\sinh\rho)^{-1}\psi_{\tau\tau}, & \psi_{\tau\theta} &= i(\sin\theta)^{-1}(\cos\theta)\psi_{\tau\tau}, \\
\psi_{\tau\phi} &= \psi_{\tau\tau}, & \psi_{\rho\rho} &= i(\cosh\rho)^{-1}(\sinh\rho)^{-1}\psi_{\tau\rho}, \\
\psi_{\rho\theta} &= i(\sin\theta)^{-1}(\cos\theta)\psi_{\tau\rho}, & \psi_{\rho\phi} &= \psi_{\tau\rho}, \\
\psi_{\theta\theta} &= i(\sin\theta)^{-1}(\cos\theta)\psi_{\tau\theta}, & \psi_{\theta\phi} &= \psi_{\tau\theta}, & \psi_{\phi\phi} &= \psi_{\tau\tau}.
\end{aligned} \tag{49}$$

The box and Laplacian action on this solution is given by

$$\square\psi_{\mu\nu} = \frac{E_0^2 - 3E_0 - 2}{L^2}\psi_{\mu\nu}, \quad \Delta\psi_{\mu\nu} = -\frac{E_0^2 - 3E_0 + 6}{L^2}\psi_{\mu\nu}. \tag{50}$$

The spin-2 massive modes were previously obtained in [20]. As pointed out in [20], acting on this highest state with $E_{-\vec{\alpha}_2}$, one obtains the $s = 1, 0, -1, -2$ modes as well. Together they form the spin-2 representation of $SO(3)$.

4 Linearized modes in AdS_5 and AdS_6

Although in the next section, we present linearized modes in AdS spacetimes in general dimensions, we nevertheless give explicit results for AdS_5 and AdS_6 in this section, since these low-lying examples are more applicable than higher-dimensional ones. In both cases, there are three Cartan generators, and hence we group these two examples together.

4.1 AdS_5

The AdS_5 metric is given by setting $u_1 = \sin\theta$, $u_2 = \cos\theta$ in the general AdS metric (5), namely

$$ds^2 = L^2 \left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho (d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) \right). \tag{51}$$

The isometry group is $SO(2, 4)$. The three Cartan generators are

$$H_1 = iL_{12} = i\frac{\partial}{\partial\tau}, \quad H_2 = iL_{34} = i\frac{\partial}{\partial\phi_1}, \quad H_3 = iL_{56} = i\frac{\partial}{\partial\phi_2}. \tag{52}$$

The simple roots are $\vec{\alpha}_1 = (-1, -1, 0)$, $\vec{\alpha}_2 = (0, 1, -1)$, $\vec{\alpha}_3 = (0, 1, 1)$. The corresponding generators are

$$\begin{aligned}
E_{\vec{\alpha}_1} &= \frac{1}{2}(L_{13} + iL_{14} + iL_{23} - L_{24}) \\
&= \frac{1}{2}e^{i(\tau+\phi_1)}\sin\theta\tanh\rho\frac{\partial}{\partial\tau} - \frac{1}{2}ie^{i(\tau+\phi_1)}\sin\theta\frac{\partial}{\partial\rho} \\
&\quad - \frac{1}{2}ie^{i(\tau+\phi_1)}\cos\theta\coth\rho\frac{\partial}{\partial\theta} + \frac{1}{2}e^{i(\tau+\phi_1)}\csc\theta\coth\rho\frac{\partial}{\partial\phi_1}, \\
E_{\vec{\alpha}_2} &= \frac{1}{2}(L_{35} + iL_{36} - iL_{45} + L_{46}) \\
&= -\frac{1}{2}e^{-i(\phi_1-\phi_2)}\frac{\partial}{\partial\theta} + \frac{1}{2}ie^{-i(\phi_1-\phi_2)}\cot\theta\frac{\partial}{\partial\phi_1} + \frac{1}{2}ie^{i(\phi_1-\phi_2)}\tan\theta\frac{\partial}{\partial\phi_2},
\end{aligned}$$

$$\begin{aligned}
E_{\vec{\alpha}_3} &= \frac{1}{2} (L_{35} - iL_{36} - iL_{45} - L_{46}) \\
&= -\frac{1}{2} e^{-i(\phi_1+\phi_2)} \frac{\partial}{\partial \theta} + \frac{1}{2} i e^{-i(\phi_1+\phi_2)} \cot \theta \frac{\partial}{\partial \phi_1} - \frac{1}{2} i e^{-i(\phi_1+\phi_2)} \tan \theta \frac{\partial}{\partial \phi_2}.
\end{aligned} \tag{53}$$

The corresponding negative-root generators are given by

$$E_{-\vec{\alpha}_1} = (E_{\vec{\alpha}_1})^*, \quad E_{-\vec{\alpha}_2} = -(E_{\vec{\alpha}_2})^*, \quad E_{-\vec{\alpha}_3} = -(E_{\vec{\alpha}_3})^*. \tag{54}$$

We shall not give the remaining generators. Note that the Cartan and simple-root generators of the $SO(4)$ subgroup of $SO(2, 4)$ are (H_2, H_3) and $(E_{\vec{\alpha}_2}, E_{\vec{\alpha}_3})$ respectively.

Scalar modes:

In this case, we must have $s_1 = s_2 = 0$, and the solution is given by

$$\Phi = e^{-iE_0\tau} (\cosh \rho)^{-E_0}. \tag{55}$$

The Laplacian action on Φ is given by

$$\Delta \Phi = -\square \Phi = -\nabla_\mu \nabla^\mu \Phi = -\frac{E_0(E_0 - 4)}{L^2} \Phi \tag{56}$$

This scalar mode is a singlet under the $SO(4)$.

Vector modes:

For our choice of Cartan generators, we must have $s_1 = 1$ and $s_2 = 0$. The solution is given by

$$\begin{aligned}
A_\tau &= e^{-i(E_0\tau+\phi_1)} \sin \theta (\cosh \rho)^{-E_0} \sinh \rho, \\
A_\rho &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} A_\tau, \\
A_\theta &= i (\sin \theta)^{-1} (\cos \theta) A_\tau, \quad A_{\phi_1} = A_\tau, \quad A_{\phi_2} = 0.
\end{aligned} \tag{57}$$

The box and Laplacian operators acting on this solution are given by

$$\square A_\mu = \frac{E_0^2 - 4E_0 - 1}{L^2} A_\mu, \quad \Delta A_\mu = -\frac{(E_0 - 1)(E_0 - 3)}{L^2} A_\mu. \tag{58}$$

Acting on this $(s_1, s_2) = (1, 0)$ vector mode with $E_{-\vec{\alpha}_2}$ and $E_{-\vec{\alpha}_3}$ repeatedly, we obtain the four-dimensional vector representation of $SO(4)$, with the weights given by $(\pm 1, 0)$ and $(0, \pm 1)$, as shown in Fig. 1.

Spin-2 modes:

For this case, we must have $s_1 = 2$ and $s_2 = 0$. The components of the solution are given by

$$\psi_{\tau\tau} = e^{-i(E_0\tau+2\phi_1)} (\sin \theta)^2 (\cosh \rho)^{-E_0} (\sinh \rho)^2,$$

$$\begin{aligned}
\psi_{\tau\rho} &= i(\cosh\rho)^{-1}(\sinh\rho)^{-1}\psi_{\tau\tau}, & \psi_{\tau\theta} &= i(\sin\theta)^{-1}(\cos\theta)\psi_{\tau\tau}, \\
\psi_{\tau\phi_1} &= \psi_{\tau\tau}, & \psi_{\tau\phi_2} &= 0, & \psi_{\rho\rho} &= i(\cosh\rho)^{-1}(\sinh\rho)^{-1}\psi_{\tau\rho}, \\
\psi_{\rho\theta} &= i(\sin\theta)^{-1}(\cos\theta)\psi_{\tau\rho}, & \psi_{\rho\phi_1} &= \psi_{\tau\rho}, & \psi_{\rho\phi_2} &= 0, \\
\psi_{\theta\theta} &= i(\sin\theta)^{-1}(\cos\theta)\psi_{\tau\theta}, & \psi_{\theta\phi_1} &= \psi_{\tau\theta}, & \psi_{\theta\phi_2} &= 0, \\
\psi_{\phi_1\phi_1} &= \psi_{\tau\phi_1}, & \psi_{\phi_1\phi_2} &= 0, & \psi_{\phi_2\phi_2} &= 0.
\end{aligned} \tag{59}$$

The box and Laplacian operators acting on this solution are given by

$$\square\psi_{\mu\nu} = \frac{E_0^2 - 4E_0 - 2}{L^2}\psi_{\mu\nu}, \quad \Delta\psi_{\mu\nu} = -\frac{E_0^2 - 4E_0 + 8}{L^2}\psi_{\mu\nu}. \tag{60}$$

Acting on this $(s_1, s_2) = (2, 0)$ spin-2 mode with $E_{-\vec{\alpha}_2}$ and $E_{-\vec{\alpha}_3}$ repeatedly, we obtain the nine-dimensional spin-2 representation of $SO(4)$, with the weights given by $(\pm 2, 0)$, $(0, \pm 2)$, $(\pm 1, \pm 1)$ and $(0, 0)$. The \pm signs are independent. In Fig. 1, we present the construction of the vector and spin-2 modes from their highest-weight states.

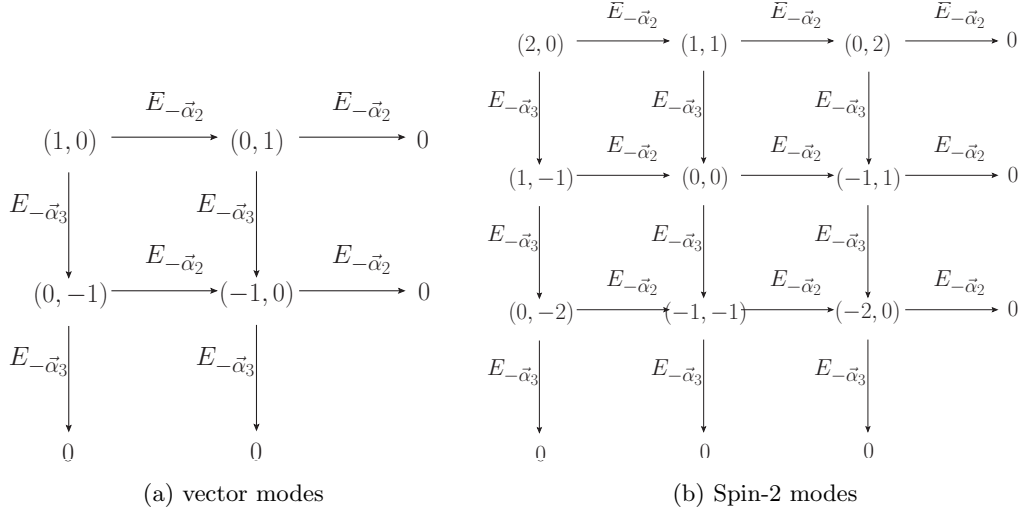


Figure 1: AdS₅ modes: We present the weights of the full vector and spin-2 representations of the $SO(4)$ subgroup of $SO(2, 4)$, and how these modes are constructed from their highest-weight states by acting with the negative-root generators.

4.2 AdS₆

The AdS₆ metric is obtained by setting $u_1 = \sin\theta_1 \sin\theta_2$, $u_2 = \sin\theta_1 \cos\theta_2$, $u_3 = \cos\theta_1$ in the general metric (5). It is given by

$$ds^2 = L^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho \left(d\theta_1^2 + \sin^2 \theta_1 \left(d\theta_2^2 + \sin^2 \theta_2 d\phi_1^2 + \cos^2 \theta_2 d\phi_2^2 \right) \right) \right). \tag{61}$$

The isometry group is $SO(2, 5)$ and the three Cartan generators are

$$H_1 = iL_{12} = i\frac{\partial}{\partial\tau}, \quad H_2 = iL_{34} = i\frac{\partial}{\partial\phi_1}, \quad H_3 = iL_{56} = i\frac{\partial}{\partial\phi_2}. \quad (62)$$

The simple roots are $\vec{\alpha}_1 = (-1, -1, 0)$, $\vec{\alpha}_2 = (0, 1, -1)$, $\vec{\alpha}_3 = (0, 0, 1)$. The corresponding generators are

$$\begin{aligned} E_{\vec{\alpha}_1} &= \frac{1}{2}(L_{13} + iL_{14} + iL_{23} - L_{24}) \\ &= \frac{1}{2}e^{i(\tau+\phi_1)} \sin\theta_1 \sin\theta_2 \tanh\rho \frac{\partial}{\partial\tau} - \frac{1}{2}ie^{i(\tau+\phi_1)} \sin\theta_1 \sin\theta_2 \frac{\partial}{\partial\rho} \\ &\quad - \frac{1}{2}ie^{i(\tau+\phi_1)} \cos\theta_1 \sin\theta_2 \coth\rho \frac{\partial}{\partial\theta_1} - \frac{1}{2}ie^{i(\tau+\phi_1)} \csc\theta_1 \cos\theta_2 \coth\rho \frac{\partial}{\partial\theta_2} \\ &\quad + \frac{1}{2}e^{i(\tau+\phi_1)} \csc\theta_1 \csc\theta_2 \coth\rho \frac{\partial}{\partial\phi_1}, \\ E_{\vec{\alpha}_2} &= \frac{1}{2}(L_{35} + iL_{36} - iL_{45} + L_{46}) \\ &= -\frac{1}{2}e^{-i(\phi_1-\phi_2)} \frac{\partial}{\partial\theta_2} + \frac{1}{2}ie^{-i(\phi_1-\phi_2)} \cot\theta_2 \frac{\partial}{\partial\phi_1} + \frac{1}{2}ie^{i(\phi_1-\phi_2)} \tan\theta_2 \frac{\partial}{\partial\phi_2}, \\ E_{\vec{\alpha}_3} &= L_{57} - iL_{67} \\ &= -e^{-i\phi_2} \cos\theta_2 \frac{\partial}{\partial\theta_1} + e^{-i\phi_2} \cot\theta_1 \sin\theta_2 \frac{\partial}{\partial\theta_2} + ie^{-i\phi_2} \cot\theta_1 \sec\theta_2 \frac{\partial}{\partial\phi_2}. \end{aligned} \quad (63)$$

The corresponding negative-root generators are given by

$$E_{-\vec{\alpha}_1} = (E_{\vec{\alpha}_1})^*, \quad E_{-\vec{\alpha}_2} = -(E_{\vec{\alpha}_2})^*, \quad E_{-\vec{\alpha}_3} = -(E_{\vec{\alpha}_3})^*. \quad (64)$$

The Cartan and the simple-root generators of the $SO(5)$ subgroup of $SO(2, 5)$ are (H_2, H_3) and $(E_{\vec{\alpha}_2}, E_{\vec{\alpha}_3})$ respectively.

Scalar modes:

The non-trivial solution arises for $s_1 = s_2 = 0$, and it is given by

$$\Phi = e^{-iE_0\tau} (\cosh\rho)^{-E_0}. \quad (65)$$

The Laplacian action on Φ is given by

$$\Delta\Phi = -\square\Phi = -\nabla_\mu \nabla^\mu \Phi = -\frac{E_0(E_0-5)}{L^2}\Phi \quad (66)$$

As in the previous examples, this mode is a singlet under the $SO(5)$.

Vector modes:

In this case, we must have $s_1 = 1$ and $s_2 = 0$, and the corresponding solution is given by

$$A_\tau = e^{-i(E_0\tau+\phi_1)} \sin\theta_1 \sin\theta_2 (\cosh\rho)^{-E_0} \sinh\rho,$$

$$\begin{aligned}
A_\rho &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} A_\tau, & A_{\theta_1} &= i (\sin \theta_1)^{-1} (\cos \theta_1) A_\tau, \\
A_{\theta_2} &= i (\sin \theta_2)^{-1} (\cos \theta_2) A_\tau, & A_{\phi_1} &= A_\tau, & A_{\phi_2} &= 0.
\end{aligned} \tag{67}$$

Note that for this solution, we have

$$\Box A_\mu = \frac{E_0^2 - 5E_0 - 1}{L^2} A_\mu, \quad \Delta A_\mu = -\frac{(E_0 - 1)(E_0 - 4)}{L^2} A_\mu. \tag{68}$$

Acting with the negative-root generators of the $SO(5)$ subgroup, we obtain the full five-dimensional spin-1 representation of $SO(5)$, with the weights $(\pm 1, 0)$, $(0, \pm 1)$ and $(0, 0)$, as shown in Fig. 2.

Spin-2 modes:

The non-trivial solution arises when $s_1 = 2$ and $s_2 = 0$, and it is given by

$$\begin{aligned}
\psi_{\tau\tau} &= e^{-i(E_0\tau+2\phi_1)} (\sin \theta_1)^2 (\sin \theta_2)^2 (\cosh \rho)^{-E_0} (\sinh \rho)^2, \\
\psi_{\tau\rho} &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} \psi_{\tau\tau}, & \psi_{\tau\theta_1} &= i (\sin \theta_1)^{-1} (\cos \theta_1) \psi_{\tau\tau}, \\
\psi_{\tau\theta_2} &= i (\sin \theta_2)^{-1} (\cos \theta_2) \psi_{\tau\tau}, & \psi_{\tau\phi_1} &= \psi_{\tau\tau}, & \psi_{\tau\phi_2} &= 0, \\
\psi_{\rho\rho} &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} \psi_{\tau\rho}, & \psi_{\rho\theta_1} &= i (\sin \theta_1)^{-1} (\cos \theta_1) \psi_{\tau\rho}, \\
\psi_{\rho\theta_2} &= i (\sin \theta_2)^{-1} (\cos \theta_2) \psi_{\tau\rho}, & \psi_{\rho\phi_1} &= \psi_{\tau\rho}, & \psi_{\rho\phi_2} &= 0, \\
\psi_{\theta_1\theta_1} &= i (\sin \theta_1)^{-1} (\cos \theta_1) \psi_{\tau\theta_1}, & \psi_{\theta_1\theta_2} &= i (\sin \theta_2)^{-1} (\cos \theta_2) \psi_{\tau\theta_1}, \\
\psi_{\theta_1\phi_1} &= \psi_{\tau\theta_1}, & \psi_{\theta_1\phi_2} &= 0, & \psi_{\theta_2\theta_2} &= i (\sin \theta_2)^{-1} (\cos \theta_2) \psi_{\tau\theta_2}, \\
\psi_{\theta_2\phi_1} &= \psi_{\tau\theta_2}, & \psi_{\theta_2\phi_2} &= 0, & \psi_{\phi_1\phi_1} &= \psi_{\tau\phi_1}, \\
\psi_{\phi_1\phi_2} &= 0, & \psi_{\phi_2\phi_2} &= 0.
\end{aligned} \tag{69}$$

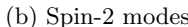
The solution satisfies

$$\Box \psi_{\mu\nu} = \frac{E_0^2 - 5E_0 - 2}{L^2} \psi_{\mu\nu}, \quad \Delta \psi_{\mu\nu} = -\frac{E_0^2 - 5E_0 + 10}{L^2} \psi_{\mu\nu}. \tag{70}$$

Acting with the negative-root generators of the $SO(5)$ subgroup, we obtain the full 14-dimensional spin-2 representation of $SO(5)$, with the weights $(\pm 2, 0)$, $(0, \pm 2)$, $(\pm 1, \pm 1)$, $(\pm 1, 0)$, $(0, \pm 1)$ and $(0, 0)_2$, where the subscript denotes the degeneracy and \pm are independent. In Fig. 2, we present the construction of the vector and spin-2 modes from their highest-weight states.

5 Linearized modes in general AdS_D

In the previous sections, we obtain explicit solutions for the linearized scalar, vector and spin-2 modes in AdS spacetimes in $D = 3, 4, 5$ and 6. The results can be easily generalized to arbitrary dimensions. In this section, we present the general solutions, which we verify up to $D = 11$.



5.1 $D = 2n + 1$

$$\begin{aligned} u_1 &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1}, \\ u_2 &= \sin \theta_1 \sin \theta_2 \cdots \cos \theta_{n-1}, \\ u_3 &= \sin \theta_1 \sin \theta_2 \cdots \cos \theta_{n-2}, \\ &\dots \\ u_n &= \cos \theta_1. \end{aligned} \tag{71}$$
$$\begin{aligned}
ds^2 = & L^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho \left(d\theta_1^2 + \cos \theta_1 d\phi_n^2 + \sin^2 \theta_1 \left(d\theta_2^2 + \cos^2 \theta_2 d\phi_{n-1}^2 \right. \right. \right. \\
& \left. \left. \left. + \sin^2 \theta_2 (\cdots (d\theta_{n-1}^2 + \cos \theta_{n-1} d\phi_2^2 + \sin \theta_{n-1} d\phi_1^2) \right) \right) \right). \quad (72)
\end{aligned}$$

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Scalar modes:

The solution requires that $s_i = 0$, and it is given by

$$\Phi = e^{-iE_0\tau} (\cosh \rho)^{-E_0}. \quad (73)$$

The Laplacian acting on Φ is

$$\Delta\Phi = -\square\Phi = -\frac{E_0(E_0 - D + 1)}{L^2}\Phi. \quad (74)$$

Vector modes:

In this case, we must have $s_1 = 1$ and $s_i = 0$ for $i \geq 2$. The solution is given by

$$\begin{aligned} A_\tau &= e^{-i(E_0\tau + \phi_1)} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1} (\cosh \rho)^{-E_0} \sinh \rho, \\ A_\rho &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} A_\tau, \quad A_{\theta_i} = i (\sin \theta_i)^{-1} (\cos \theta_i) A_\tau, \\ A_{\phi_1} &= A_\tau, \quad A_{\phi_2} = \cdots = A_{\phi_n} = 0. \end{aligned} \quad (75)$$

It satisfies

$$\square A_\mu = \frac{E_0^2 - (D-1)E_0 - 1}{L^2} A_\mu, \quad \Delta A_\mu = -\frac{(E_0 - 1)(E_0 - (D-2))}{L^2} A_\mu. \quad (76)$$

Spin-2 modes:

In this case, we have $s_1 = 2$ and $s_i = 0$ for $i \geq 2$. The solution is given by

$$\begin{aligned} \psi_{\tau\tau} &= e^{-i(E_0\tau + 2\phi_1)} (\sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^2 (\cosh \rho)^{-E_0} (\sinh \rho)^2, \\ \psi_{\tau\rho} &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} \psi_{\tau\tau}, \quad \psi_{\tau\theta_i} = i (\sin \theta_i)^{-1} (\cos \theta_i) \psi_{\tau\tau}, \\ \psi_{\tau\phi_1} &= \psi_{\tau\tau}, \quad \psi_{\tau\phi_2} = \cdots = \psi_{\tau\phi_n} = 0, \\ \psi_{\rho\rho} &= i (\cosh \rho)^{-1} (\sinh \rho)^{-1} \psi_{\tau\rho}, \quad \psi_{\rho\theta_i} = i (\sin \theta_i)^{-1} (\cos \theta_i) \psi_{\tau\rho}, \\ \psi_{\rho\phi_1} &= \psi_{\tau\rho}, \quad \psi_{\rho\phi_2} = \cdots = \psi_{\rho\phi_n} = 0, \\ \psi_{\theta_i\theta_i} &= i (\sin \theta_i)^{-1} (\cos \theta_i) \psi_{\tau\theta_i}, \quad \psi_{\theta_i\theta_j} = i (\sin \theta_j)^{-1} (\cos \theta_j) \psi_{\tau\theta_i}, \\ \psi_{\theta_i\phi_1} &= \psi_{\tau\theta_i}, \quad \psi_{\theta_i\phi_2} = \cdots = \psi_{\theta_i\phi_n} = 0, \quad \psi_{\phi_1\phi_1} = \psi_{\tau\phi_1}, \\ \psi_{\phi_1\phi_2} &= \cdots = \psi_{\phi_1\phi_n} = \psi_{\phi_2\phi_2} = \cdots = \psi_{\phi_2\phi_n} = \cdots = \psi_{\phi_n\phi_n} = 0. \end{aligned} \quad (77)$$

The box and Laplacian action on this solution is given by

$$\square\psi_{\mu\nu} = \frac{E_0^2 - (D-1)E_0 - 2}{L^2} \psi_{\mu\nu}, \quad \Delta\psi_{\mu\nu} = -\frac{E_0^2 - (D-1)E_0 + 2(D-1)}{L^2} \psi_{\mu\nu}. \quad (78)$$

5.2 $D = 2n$

The AdS_{2n} metric is given by (5) with u_i also given by (71). To be explicit, we have

$$ds^2 = L^2 \left(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho \left(d\theta_1^2 + \sin^2 \theta_1 \left(d\theta_2^2 + \cos^2 \theta_2 d\phi_{n-1}^2 \right. \right. \right. \\ \left. \left. \left. + \sin^2 \theta_2 (\cdots (d\theta_{n-1}^2 + \cos \theta_{n-1} d\phi_2^2 + \sin \theta_{n-1} d\phi_1^2) \right) \right) \right). \quad (79)$$

Comparing to the AdS_{2n+1} metric, there are no ϕ_n coordinate in AdS_{2n} . Thus the scalar modes take the identical form as (73). The vector and spin-2 modes can also be read off from those in odd dimensions (75) and (77) respectively, by ignoring the ϕ_n components, which vanish anyway. The box and Laplacian action on these modes are given by (74), (76) and (78), but with $D = 2n$.

Having obtained the highest weight states, we can obtain the remaining modes in the representation of the subgroup $SO(D-1)$ of $SO(2, D-1)$. The scalar mode is a singlet under the $SO(D-1)$. The vector modes form $(D-1)$ -dimensional spin-1 representation. For $D = 2n+1$, the weights are n -vectors, $(0, \dots, \pm 1, \dots, 0)$. For $D = 2n+2$, an extra weight vector arises, namely $(0, \dots, 0)$. The spin-2 modes, on the other hand, form the spin-2 representation of $SO(D-1)$ with dimensions $\frac{1}{2}D(D-1) - 1$. For $D = 2n+1$, the weights are $(0, \dots, \pm 2, \dots, 0)$, $(0, \dots, \pm 1, \dots, \pm 1, \dots, 0)$ and $(0, \dots, 0)_{n-1}$, where \pm are independent and the subscript denotes the degeneracy. In $D = 2n+2$, additional weights, $(0, \dots, \pm 1, \dots, 0)$ and $(0, \dots, 0)$ arise.

6 Log modes

In higher-derivative extended gravities, for appropriate choice of parameters, the differential operator may factorize into the follow form

$$(\tilde{\square} - m_1^2) \cdots (\tilde{\square} - m_n^2) \psi = 0. \quad (80)$$

where $\tilde{\square} \equiv \square + c\Lambda$ for appropriate constant c and $\tilde{\square}\psi_0 = 0$ define a massless mode ψ_0 . (See, for example, [6, 8].) Thus the most general solution is the linear combination of all the modes associated with mass m_n . A critical point is defined when two mass parameters become coincident, in which case log modes emerge. The log modes are defined by

$$(\tilde{\square} - m_1^2)^2 \psi_{\log} = 0, \quad \text{but} \quad (\tilde{\square} - m_1^2)_{\log} \neq 0. \quad (81)$$

From the general massive solutions we obtained in the previous section, we can easily obtain the log modes, namely

$$\psi_{\log} = \frac{\partial\psi(m^2)}{\partial(m^2)} \Big|_{m \rightarrow m_1}. \quad (82)$$

Since all the solutions have a universal factor $e^{-iE_0\tau}(\cosh\rho)^{-E_0}$, it is easy to see that the log modes are proportional to the corresponding massive or massless modes with an overall factor

$$f = -i\tau - \log(\cosh\rho). \quad (83)$$

The log modes were obtained in [29] in three dimensions and [20] in four dimensions. When there are multiple m_i that are coincident, so that the equation becomes

$$(\tilde{\square} - m_1)^k \psi = 0. \quad (84)$$

The most general solution of (84) is then given by

$$\psi = \sum_{i=0}^{k-1} c_i \frac{\partial^i \psi(m^2)}{\partial(m^2)^i} \Big|_{m \rightarrow m_1} = \left(\sum_{i=0}^{k-1} c_i f^i \right) \psi(m_1^2). \quad (85)$$

7 Conclusions and discussions

In this paper, we have constructed the general massless and massive scalar, vector and spin-2 modes in AdS vacua in diverse dimensions. These modes may arise in extended and critical gravities with higher-order curvature terms. As we have mentioned in the introduction, the explicit solutions may yield useful information of the relevant theories. An important property of these modes is the following constraint

$$M_s^2 = \frac{E_0^2 - (D-1)E_0}{L^2}, \quad (86)$$

where M_s^2 , defined in (22), denotes M_0^2 , M_1^2 and M_2^2 that appear in (19). The reality condition for E_0 , which ensures that there is no exponential growth in time, is given by

$$M_s^2 \geq M_{\text{BF}}^2 \equiv -\frac{(D-1)^2}{4L^2}. \quad (87)$$

Thus, we see that there is a universal Breitenlohner-Freedman bound for the scalar, vector and the spin-2 fields. We expect that this result generalizes to higher spin- s fields. Note that in the case of topologically massive gravity, the operator is factorized to three linear factors with no square root parameters, and hence there is no such a bound.

As we have explained in section 2, the parameter M_s does not necessarily denote the true mass of the modes. In order to find out the proper definition of the mass, we need to

define the meaning of masslessness in the AdS spacetimes. Let $r = \sinh \rho$ denote the radius of the foliating sphere in the AdS given in (5), we find that for large r , the spin- s modes we obtained fall off as follows

$$\psi_s \sim \frac{1}{r^{E_0-s}}. \quad (88)$$

It is natural to expect that all the massless modes should fall off the same way as the Schwarzschild-AdS black holes. Thus for massless modes we have

$$E_0 = D - 3 + s. \quad (89)$$

It follows that a massless spin- s mode $\psi_s^{(0)}$ satisfies

$$\left(\square + \frac{2 - (s-2)(D+s-4)}{L^2} \right) \psi_s^{(0)} = 0. \quad (90)$$

The corresponding parameter M_s^2 is given by

$$M_s^2 = (M_s^{(0)})^2 \equiv \frac{(s-2)(D+s-3)}{L^2} = M_{\text{BF}}^2 + \frac{(D+2s-5)^2}{L^2} \geq M_{\text{BF}}^2. \quad (91)$$

We see that the massless modes always satisfy the tachyon-free condition. Note that this definition of massless fields coincides precisely the one discussed in section 2 for $s = 1, 2$. For the scalar $s = 0$ field, it follows from (91) that the massless mode is given by $(M_0^{(0)})^2 = -2(D-3)$. We verify, with many examples of supergravities (*e.g.*, [28],) that this definition of massless scalar is consistent with supergravity scalar potentials. (Scalars that belong to the graviton super-multiplet in supergravities are naturally massless.) It follows that the true mass of the spin- s solutions we obtained is given by

$$\mathcal{M}_s^2 = M_s^2 - (M_s^{(0)})^2 = \frac{(E_0 - D + 3 - s)(E_0 - 2 + s)}{L^2}. \quad (92)$$

The true mass \mathcal{M}_s of a spin- s field in the AdS background can thus be defined by

$$\left(\square + \frac{2 - (s-2)(D+s-4)}{L^2} - \mathcal{M}_s^2 \right) \psi_s = 0. \quad (93)$$

In terms of this properly defined mass \mathcal{M}_s , the generalized Breitenlohner-Freedman bound is now given by

$$(\mathcal{M}_s^{\text{BF}})^2 \geq -\frac{(D+2s-5)^2}{4L^2}. \quad (94)$$

Having obtained the generalized Breitenlohner-Freedman bound that ensures tachyon free, we examine the falloffs of the modes at the AdS boundary. It follows from (86) and (89) that if $D-5+2s \leq 0$, the modes ψ_s have falloffs no slower than the Schwarzschild black hole at the AdS boundary. For modes with $D-5+2s > 0$, which include the spin-2 modes,

two situations can occur. The first-type of modes have $M_s^2 \geq (M_s^{(0)})^2$, corresponding to $E_0 \geq D-3+s$, they satisfy the standard AdS boundary condition. (The $E_0 \leq 2-s$ branch violates the boundary condition.) The second-type of modes have M_s^2 lie in the range,

$$M_{\text{BF}}^2 \leq M_s^2 < (M_s^{(0)})^2. \quad (95)$$

They have slower falloffs than the standard AdS boundary condition. These modes should be truncated out from the spectrum. Our explicit construction of the linearized modes confirms the conclusion of [8].

To conclude, the explicit construction of the linearized modes that could arise in extended and critical gravities with higher-derivative curvature terms should be served as useful tool to study various properties of these theories.

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